B.Sc Physics (Hons), Part I
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Ques Applying Fourier's theorem to analysis of a saw tooth wave into its Harmonic components.

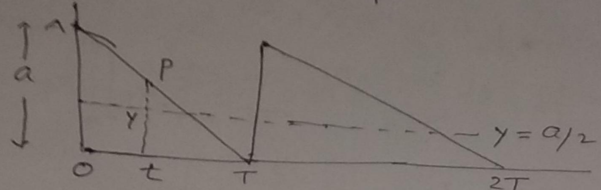
Ans:-

Analysis of Saw-Tooth wave :→

The displacement of a complex vibration of saw-tooth wave is given by

$y = a$ at $T = 0$ and

$y = 0$ at $t = T$



The displacement at any instant 't' may be given as

$$y = \cancel{a\sin}\ a\left(1-\frac{t}{T}\right) \quad \text{for } 0<t<T \quad \cdots (1)$$

when the time axis passes through the lowest points of the displacement curve,

Eq^n (1) can be derived from similar triangles

[OAT and tPT. since $\frac{a}{T} = \frac{y}{T-t}$]

For Fourier's series is

$$y = f(t) = A_0 + A_1\cos\omega t + \cdots + A_r\cos r\omega t + \cdots$$
$$\cdots + B_1\sin\omega t + \cdots + B_r\sin r\omega t + \cdots \quad (2)$$

where $A_0 = \frac{1}{T}\int_0^T y\,dt$

$$A_r = \frac{2}{T}\int_0^T y\cos r\omega t\,dt$$

and $B_r = \frac{2}{T}\int_0^T y\sin r\omega t\,dt$

Now, $A_0 = \frac{1}{T}\int_0^T y\,dt$

$$= \frac{a}{T}\int_0^T \left(1-\frac{t}{T}\right)dt$$

$$= \frac{a}{T}\left[1-\frac{t^2}{2T}\right]_0^T$$

$$= a/2$$

For A_r, $A_r = \frac{2}{T}\int_0^T y\cos r\omega t\,dt$

$$= \frac{2a}{T}\int_0^T\left(1-\frac{t}{T}\right)\cos r\omega t\,dt$$

$$= \frac{2a}{T}\int_0^T \frac{2\pi rt}{T}\,dt - \frac{2a}{T^2}\int_0^T t\cdot\cos\frac{2\pi rt}{T}\,dt$$

$$= -\frac{2a}{T}\left[t\cdot\cos\frac{2\pi r}{T}t\cdot dt \quad \left[\because \int_0^T \cos\frac{2\pi r}{T}t\,dt = 0\right]\right.$$

$$= -\frac{2a}{T^2}\left[t\cdot\frac{\sin\frac{2\pi rt}{T}}{\frac{2r\pi}{T}} + \frac{\cos\frac{2\pi r}{T}t}{(2\pi r/T)^2}\right]_0^T$$

$$= -\frac{2a}{T^2}\left[\frac{1}{\left(\frac{2\pi r}{T^2}\right)^2} - \frac{1}{\left(\frac{2\pi r}{T}\right)^2}\right]$$

$$= 0 \qquad [\because \sin 2r\pi = 0 \ \&\ \cos 2r\pi = 1]$$

Similarly $B_r = \frac{2}{T}\int_0^T a\left(1-\frac{t}{T}\right)\sin\frac{2r\pi}{T}t\cdot dt$

$$= \frac{2a}{T}\int_0^T \sin\frac{2r\pi}{T}t\,dt - \frac{2a}{T^2}\int_0^T t\sin\frac{2\pi r}{T}t\cdot dt$$

$$= -\frac{2a}{T^2}\int_0^T t\sin\frac{2r\pi}{T}t\,dt \qquad \left[\because \int_0^T \sin\frac{2\pi r}{T}t\,dt = 0\right]$$

$$= -\frac{2a}{T^2}\left[t\cdot\frac{\cos\frac{2r\pi}{T}t}{2r\pi/T} - \frac{\sin\frac{2\pi r}{T}t}{(2\pi r/T)^2}\right]_0^T$$

$$= \frac{2a}{T^2}\left[T\cdot\frac{\cos 2r\pi}{2r\pi/T} - \frac{\sin 2r\pi}{(2r\pi/T)^2} - 0 + \frac{\sin 0}{(2r\pi/T)^2}\right]$$

$$= \frac{2a}{T^2}\left[T\cdot\frac{1}{2\pi r/T}\right] \qquad \left[\because \cos 2\pi r = 1 \ \&\ \sin 2\pi r = 0\right]$$

$$\therefore B_r = \frac{a}{2\pi}$$

Thus $B_1 = \frac{a}{\pi}$, $B_2 = \frac{a}{2\pi}$, $B_3 = \frac{a}{3\pi}$ .... (3)

Substituting the value of A_0, A_r & B_r, we get

$$y = \frac{a}{2} + \frac{a}{\pi}\left(\sin\frac{2\pi}{T}t + \frac{1}{2}\sin\frac{4\pi}{T}t + \frac{1}{3}\sin\frac{6\pi}{T}t + \cdots\right]$$

Thus the given periodic motion has the axis $y = \frac{a}{2}$ above the time axis and the harmonic components

$\frac{a}{\pi}\sin\frac{2\pi}{T}t$, $\frac{a}{2\pi}\sin\frac{4\pi}{T}t$, $\frac{a}{3\pi}\sin\frac{6\pi}{T}t$ ... and so on, having frequencies in the ratio $1:2:3$ ... and amplitudes in ratio $1:\frac{1}{2}:\frac{1}{3}:$ .....